An introduction on hyper-bag-graphs and their application to hypergraph e-adjacency tensor

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Two talks in one

Plan

- **Part I: An introduction on hyper-bag-graphs**
- **Part II: An application of hb-graphs to general hypergraph e-adjacency tensor**

Focusing on hb-graphs

Plan

- **Part I: An introduction on hyper-bag-graphs**
- **Part II: An application of hb-graphs to general hypergraph e-adjacency tensor**

Why hb-graphs?

Research context

- PhD started in 10.2016 @ University of Geneva
  Hypergraph Modeling and Visualisation of Complex Collaboration Networks
- Done within the **Collaboration Spotting** project @ CERN
  => enhancing co-occurences in datasets

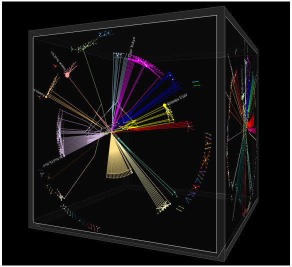


Figure 1: DataHyperCube: prototype in Ouvrard et al. [2018b]

In project...

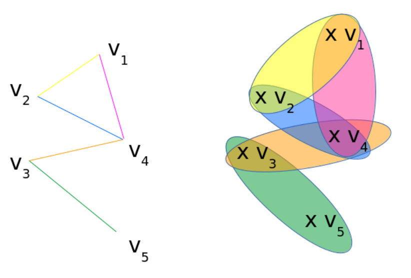
- **Datasets** modeled and stored as **labelled graphs**.
- **Co-occurences** through a **reference**.
- Multiple **facets** of dataset can be visualized.

but co-occurences are...

- Bags of elements
- n**-adic relationships**
- if bags reduced to sets:
  $\rightsquigarrow$ **hypergraphs** well fitted to model it!
- otherwise we need families of multisets
  $\rightsquigarrow$ **hb-graphs** are needed

Hypergraphs (as a reminder)

From graphs to hypergraphs



- **Hypergraphs** $\equiv$ generalisation of graphs to multiple node links
- Hypergraphs introduced by Berge and Minieka [1973].

Definition

Bretto [2013]:
A **hypergraph** $\mathcal{H}$ family of subsets of a vertex set

Elements of family $\rightsquigarrow$ **hyperedges**.

Two visions

- **set of elements** of power set of nodes $\rightsquigarrow$ set view
- **extension of graphs** $\rightsquigarrow$ n-adic relationship view

Multisets I

Definitions

- **Multiset**: a universe and a multiplicity function $A_m = (A, m)$
- **Natural multiset**: the range of the multiplicity function is a subset of $\mathbb{N}$.
- In natural multisets: two views:
  - weighted set: $A_m = \left\{x_1^{m_1}, \ldots, x_n^{m_n}\right\}$
  - collection of objects
    $$\left\{\left\{\underbrace{x_1, \ldots, x_1}_{m_1 \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{m_n \text{ times}}\right\}\right\}$$
- **Support** of the multiset $A_m^\star$:
  $$A_m^\star = \{x \in A \colon m(x) \neq 0\}$$
- **m-cardinality** of a multiset A_m:
  $$\#_m A_m = \sum_{x \in A} m(x).$$

Multisets and operations

Let $\mathcal{A} = U_{m_\mathcal{A}}$ and $\mathcal{B} = U_{m_\mathcal{B}}$ be two msets on the same universe U.

- $\mathcal{A}$ is **included** in $\mathcal{B}$ ($\mathcal{A} \subseteq \mathcal{B}$) if $\forall x \in U$: $m_\mathcal{A}(x) \leqslant m_\mathcal{B}(x)$.
  In this case: $\mathcal{A}$ is a **submset** of $\mathcal{B}$.
- **Union** of $\mathcal{A}$ and $\mathcal{B}$: mset $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, universe U,
  $$\forall x \in U : m_\mathcal{C}(x) = \max\left(m_\mathcal{A}(x), m_\mathcal{B}(x)\right).$$
- **Intersection** of $\mathcal{A}$ and $\mathcal{B}$: mset $\mathcal{D} = \mathcal{A} \cap \mathcal{B}$, universe U,
  $$\forall x \in U : m_\mathcal{D}(x) = \min\left(m_\mathcal{A}(x), m_\mathcal{B}(x)\right).$$
- **Sum of $\mathcal{A}$ and $\mathcal{B}$**: mset $\mathcal{E} = \mathcal{A} \uplus \mathcal{B}$, universe U,
  $$\forall x \in U : m_\mathcal{E}(x) = m_\mathcal{A}(x) + m_\mathcal{B}(x).$$
- **Power multiset** of $\mathcal{A}$: set $\widetilde{\mathcal{P}}(\mathcal{A})$ of all submsets of $\mathcal{A}$.
- More in Singh et al. [2007].

Multisets II

Vector representation

Given: a natural multiset $A_m = (A, m)$ of universe $A = \{\alpha_i : i \in \llbracket n \rrbracket\}$ and multiplicity function m. It yields:

$$A_m = \left\{ \alpha_{i_j}^{m\left(\alpha_{i_j}\right)} : \alpha_{i_j} \in A_m^\star \right\}.$$

- **Vector representation**: $\overrightarrow{A} = (m(\alpha))_{\alpha \in A}$.
- Sum of the elements of $\overrightarrow{A}$: $\sharp_m A_m$
- $|A|$ elements to be described but only $|A_m^\star|$ are non-zero
- => useful for building incidence matrix of hb-graphs

Multisets III

Given: a natural multiset $A_m = (A, m)$ of universe $A = \{\alpha_i : i \in [\![n]\!]\}$ and multiplicity function m. It yields:

$$A_m = \left\{ \alpha_{i_j}^{m\left(\alpha_{i_j}\right)} : \alpha_{i_j} \in A_m^\star \right\}.$$

Unnormalised hypermatrix representation

- $A_u = \left(a_{u,i_1 \dots i_r}\right)_{i_1,\dots,i_r \in [\![n]\!]}$, symmetric, order: $r = \sharp_m A_m$, dimension n
- $a_{u,i_1 \dots i_r} = 1$ if $\forall j \in [\![r]\!] : i_j \in [\![n]\!] \wedge \alpha_{i_j} \in A_m^\star$.
- Other elements are zero.
- $|\{a_{u,i_1 \dots i_r} \neq 0, i_1, \dots, i_r \in [\![n]\!]\}| = \dfrac{r!}{\prod\limits_{\alpha \in A_m^\star} m(\alpha)}$
- n^r elements but only one needed
- $\sum\limits_{i_1,\dots,i_r \in [\![n]\!]} a_{u,i_1 \dots i_r} = \dfrac{r!}{\prod\limits_{\alpha \in A_m^\star} m(\alpha)}$.

Normalised hypermatrix representation

- $A = \left(a_{i_1 \dots i_r}\right)_{i_1,\dots,i_r \in [\![n]\!]}$, symmetric, order $r = \sharp_m A_m$, dimension n
- $a_{i_1 \dots i_r} = \dfrac{\prod\limits_{\alpha \in A_m^\star} m(\alpha)}{(r-1)!}$ if $\forall j \in [\![r]\!] : i_j \in [\![n]\!] \wedge \alpha_{i_j} \in A_m^\star$.
- Other elements are equal to zero.
- $|\{a_{u,i_1 \dots i_r} \neq 0, i_1, \dots, i_r \in [\![n]\!]\}| = \dfrac{r!}{\prod\limits_{\alpha \in A_m^\star} m(\alpha)}$
- n^r elements but only one needed
- $\sum\limits_{i_1,\dots,i_r \in [\![n]\!]} a_{i_1 \dots i_r} = r$.

Hb-graphs I

Hyper-Bag-graph or hb-graph

- **Hb-graph $\mathcal{H} = (V, E)$**: family of multisets $E = (e_i)_{i \in I}$[2]- called **hb-edges** - where the hb-edges have:
    - same universe $V = \{v_1, \ldots, v_n\}$, called **vertex set**.
    - support a subset of V.
    - each hb-edge has its own multiplicity function $m_e : V \to \mathbb{W}$ where $\mathbb{W} \subset \mathbb{R}^+$.
- **Hb-graph with no repeated hb-edge**:

$$\forall i_1 \in I, \forall i_2 \in I : e_{i_1} = e_{i_2} \Rightarrow i_1 = i_2$$

- **Order** of a hb-graph $\mathcal{H}$: $O(\mathcal{H}) = \sum_{j=1}^{n} \max_{e \in E} (m_e(v_j))$.
- **Size** of a hb-graph: $|E|$.
- **Natural hb-graph**: when all multiplicity functions have their range included in $\mathbb{N}$

[2] $I = [\![p]\!]$

Hb-graphs II

Hyper-Bag-graph or hb-graph

- **Support hypergraph** $\underline{\mathcal{H}}$: hypergraph of the support of the multisets
- **Star of a vertex**: $\forall x \in V : H(x) = \left\{ e_i^{m_{e_i}(x)} : e_i \in E \wedge x \in e_i^* \right\}$.
- **m-degree of a vertex**: $\deg_m(x) = \#_m H(x)$.

m-range, m-co-range

- **range** of a hb-graph: $r(\mathcal{H})$: range of its support hypergraph $\underline{\mathcal{H}}$.
- **m-range** of a hb-graph: $r_m(\mathcal{H}) = \max\limits_{e \in E} \#_m e$.
- **co-range** of a hb-graph: $r(\mathcal{H})$: co-range of its support hypergraph $\underline{\mathcal{H}}$.
- **m-co-range** of a hb-graph: $cr_m(\mathcal{H}) = \min\limits_{e \in E} \#_m e$.

Hb-graphs III

Particular cases

- k**-m-uniform** hb-graph: all its hb-edges of same m-cardinality k.
- k**-uniform** hb-graph: support hypergraph is k-uniform.
- A hb-graph $\mathcal{H}$ is k-m-uniform if and only if $r_m(\mathcal{H}) = cr_m(\mathcal{H}) = k$.
- A hb-graph $\mathcal{H}$ is k-uniform if and only if $r(\mathcal{H}) = cr(\mathcal{H}) = k$.
- A **hypergraph** can be seen as a natural hb-graph with multiplicity function ranges in $\{0, 1\}$

Hb-graphs IV

Sum of two hb-graphs

Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two hb-graphs.

- $V_1 \cup V_2$ as vertex set
- $E_1 + E_2$ as hb-edge family: hb-edges are obtained from the hb-edges of E_1 and E_2 with same multiplicity for vertices of V_1 (respectively V_2) but such that for each hyperedge in E_1 (respectively E_2) the universe is extended to $V_1 \cup V_2$ and the multiplicity function is extended such that $\forall v \in V_2 \backslash V_1 : m(v) = 0$ (respectively $\forall v \in V_1 \backslash V_2 : m(v) = 0$)
- $\boldsymbol{\mathcal{H}_1 + \mathcal{H}_2 = (V_1 \cup V_2, E_1 + E_2)}$

Direct sum

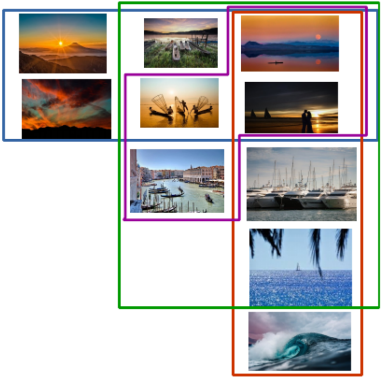
- If $E_1 + E_2$ doesn't contain any new pair of repeated hb-edge than the ones already existing in E_1 and those already existing in E_2: we have a direct sum
- In this case the sum is written $\boldsymbol{\mathcal{H}_1 \oplus \mathcal{H}_2}$.

Incidence matrix of a hb-graph

Incidence

- hb-edges are incident if their intersection is not empty
- **Incidence matrix of the hb-graph** $\mathcal{H}$: $H = [m_j\,(v_i)]_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant p}}$.
- Used in: diffusion by exchange in Ouvrard et al. [2018c]
- Incidence is a pairwise concept: a vertex is incident to a hb-edge.
- The rows allow to see which hb-edges are incident: linked by rows.

Hb-graphs: extending hypergraphs



Photos from https://www.pexels.com/photo/sailboats-racing-163318/

Slide presented at CBMI 2018 La Rochelle

	sunset	boat	ocean	person
	1	0	0	0
	1	0	0	0
	1	4	0	0
	1	3	0	3
	1	1	1	1
	1	2	1	2
	0	28	0	17
	0	1	1	0
	0	0	1	0

Paths in hb-graphs

Numbering copies

- **Strict m-path** from a vertex x to a vertex y:
  - vertex / hb-edge alternation: $x_0 e_1 x_1 \dots e_s x_s$
  - $x_0 = x$, $x_s = y$, $x \in e_1$ and $y \in e_s$ and that for all $i \in [\![s-1]\!]$, $x_i \in e_i \cap e_{i+1}$.
  - number of possible strict m-path: $m_{e_1}(x_0) \prod\limits_{i \in [\![s-1]\!]} m_{e_i \cap e_{i+1}}(x_i)\, m_{e_s}(x_s)$
- **Large m-path** from a vertex x to a vertex y:
  - same conditions but $\forall i \in [\![s-1]\!]$, $x_i \in e_i \cup e_{i+1}$.
  - number of possible large m-path: $m_{e_1}(x_0) \prod\limits_{i \in [\![s-1]\!]} m_{e_i \cup e_{i+1}}(x_i)\, m_{e_s}(x_s)$
- **Length of a path** $l(x, y) = s$
- **Almost cycle:** m-path with extremities that are different copies of the same object
- **Cycle**: m-path with extremities that are same copies of the object
- **Distance** from x to y:
  - minimal length $d(x, y)$ of an m-path from x to y if such an m-path exists.
  - If no m-path exist, x and y are said disconnected and $d(x, y) = +\infty$.
- Notion of **connected hb-graph** related to the connection of its support hypergraph

How are hb-graphs useful?

Visualisation of exchange-based diffusion

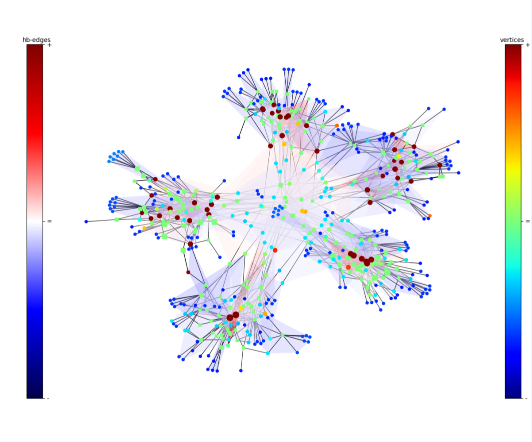


Figure 2: From Ouvrard et al. [2018c] © IEEE 2018

Applications

- Diffusion in hb-graphs and RW => see Ouvrard et al. [2018c]
- e-adjacency hypermatrix of hypergraphs => see next part

On example

- 548 vertices
- 300 hb-edges
- 5 groups
- 10 vertices in between the 5 groups

Focusing on e-adjacency tensor

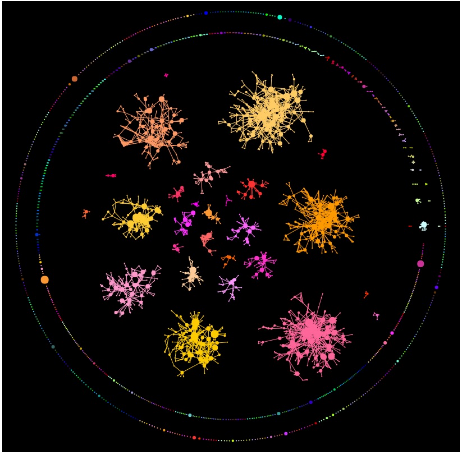
Plan

- Part I: An introduction on hyper-bag-graphs
  => Do you have any question so far?
- **Part II: An application of hb-graphs to general hypergraph e-adjacency tensor**

Why e-adjacency tensor?

Ideas behind

- Ranking of vertices in graphs
  => **random walks**
- RW for hypergraphs exist
- Diffusion = local process
  => knowledge of the neighbourhood.
- Study of diffusion process => **Laplacian**
- Incidence and adjacency matrices keep only pairwise information
- Pairwise adjacency is too restrictive for hypergraphs
- Higher order adjacency requires tensor
- Laplacian tensor is linked to the adjacency tensor
- Adjacency tensor for uniform hypergraph is known Cooper and Dutle [2012]



On adjacency

In graphs

- Two vertices are said **adjacent** if it exists an edge linking them
  => **pairwise relationship**
- Vertices incident to one given edge are said **e-adjacent**.
  => also **pairwise relationship**
- **e-adjacency and adjacency are equivalent in graphs**

Extending to hypergraphs

- k vertices are said **k-adjacent** if it exists a hyperedge that hold them
  => multi-adic relationship
- Vertices of a given hyperedge are said to be **e-adjacent**.
  => multi-adic relationship
- $\overline{k}$-adjacency: maximal k-adjacency that can be found in a given hypergraph
- In k-uniform hypergraph:
  - $\overline{k}$-adjacency is k-adjacency
  - Equivalence $\overline{k}$-adjacency and e-adjacency.
- In general hypergraphs: the equivalence doesn't hold anymore!

Tensor for general hypergraphs: the art of filling

What about this?

=> **We need to store additional information**

Existing e-adjacency tensor of a hypergraph I

Symmetric e-adjacency tensor[3]

Let $\mathcal{H} = (V, E)$ with $V = \{v_1 , v_2, \ldots , v_n\}$ and family $E = \{e_1, e_2, \ldots, e_p\}$.
Let $k_{\max} = \max\{|e_i| : e_i \in E\}$ be the maximum cardinality of the family of hyperedges.
The ([Author's note]: **e-**)**adjacency hypermatrix** of $\mathcal{H}$ written
$\mathcal{A}_{\mathcal{H}} = \left(a_{i_1 \ldots i_{k_{\max}}}\right)_{1 \leqslant i_1, \ldots, i_{k_{\max}} \leqslant n}$ is such that for a hyperedge: $e = \left\{v_{l_1}, \ldots, v_{l_s}\right\}$ of cardinality $s \leqslant k_{\max}$.

$$a_{p_1 \ldots p_{k_{\max}}} = \frac{s}{\alpha}, \text{ where } \alpha = \sum_{\substack{k_1, \ldots, k_s \geqslant 1 \\ \sum k_i = k_{max}}} \frac{k_{\max}!}{k_1! \ldots k_s!}$$

with $p_1, \ldots, p_{k_{\max}}$ chosen in all possible way from $\{l_1, \ldots, l_s\}$ with at least once from each element of $\{l_1, \ldots, l_s\}$.

[3]Banerjee et al. [2017]

Existing e-adjacency tensor of a hypergraph II

Layered e-adjacency tensor[4]

Let $\mathcal{H} = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$ and family $E = \{e_1, e_2, \dots, e_p\}$.
Let $k_{\max} = \max\{|e_i| : e_i \in E\}$ be the maximum cardinality of the family of hyperedges.
The **layered e-adjacency hypermatrix** of $\mathcal{H}$ written $\mathcal{A}_{\mathcal{H}} = \left(a_{i_1 \dots i_{k_{\max}}}\right)_{1 \leqslant i_1, \dots, i_{k_{\max}} \leqslant n}$ is such that for each hyperedge: $e = \left\{v_{l_1}, \dots, v_{l_s}\right\}$ of cardinality $s < k_{\max}$ it is completed in a hyperedge $\overline{e} = \left\{v_{l_1}, \dots, v_{l_s}, y_s, \dots, y_{k_{\max}-1}\right\}$.

$$a_{\sigma(l_1)\dots\sigma(l_s)\sigma(n+s)\dots\sigma(n+k_{\max}-1)} = \frac{1}{(k_{\max}-1)!}$$

where $\sigma \in \mathcal{S}_{k_{\max}}$.

[4] Ouvrard et al. [2017, 2018a]

Existing e-adjacency tensor of a hypergraph III

Prefix e-adjacency tensor[5]

The ([Author's note]: **prefix e-)adjacency hypermatrix** of a general hypergraph $\mathcal{H} = (V, E)$, with V as vertex set ([Author's note]: identified to $[\![n]\!]$) and E as hyperedge set, having range $r(\mathcal{H}) = k_{\text{max}}$ is an order k_{max} and dimension $|V|$ hypermatrix with entries:

$$a_{i_1 \dots i_{k_{\text{max}}}} = \begin{cases} \dfrac{1}{(s-1)!} & i_1 = \dots = i_{k-s+1}, \{i_{k-s+1}, \dots, i_k\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

[5] Sun et al. [2018] New Published 27.09.2018

Why alternative proposals?

Motivations

- The e-adjacency tensor should be easily interpretable:
  - in term of e- and $\overline{k}$-adjacency;
  - in term of the process used to build it.
- There is not a unique way of filling: additional ways require multisets for interpretability => Find other proposals hopefully easier to analyse spectrally.
- Symmetry is a desirable property

Key points of our contribution

- Hb-graphs as extension of hypergraphs
- k-adjacency tensor of a m-uniform hb-graph
- Three e-adjacency tensors in hb-graphs
- Final choice for the e-adjacency tensor

Elementary hb-graphs

Elementary hb-graph and its $\overline{k}$-adjacency hypermatrix

- **Elementary hb-graph**: hb-graph with only one non repeated hb-edge in its hb-edge family. Typically: $\mathcal{H}_e = (V, (e))$.
- For $\mathcal{H}_e = (V, (e))$:
  - e described uniquely by its hypermatrix representation Q_e.
  - $\mathcal{H}_e$ also uniquely described by Q_e.
- Let $e = \left\{ v_{j_1}^{m_{j_1}}, \ldots, v_{j_k}^{m_{j_k}} \right\}$ be a hb-edge of $\mathcal{H}$ of m-rank r.
  Normalised $\overline{k}$-adjacency hypermatrix of an elementary hb-graph $\mathcal{H}_e$ is the normalised representation of the multiset e:
  - symmetric hypermatrix $Q_e = (q_{i_1 \ldots i_r})$ of rank r and dimension n
  - only nonzero elements are:

$$q_{\sigma(j_1)^{m_{i\,\sigma(j_1)}}, \ldots, \sigma\left(j_{k_i}\right)^{m_{i\,\sigma\left(j_{k_i}\right)}}} = \frac{m_{ij_1}! \ldots m_{ij_{k_i}}!}{(r-1)!}$$

  where $\sigma \in \mathcal{S}_{[\![r]\!]}$.
- **Claim:** Let $\mathcal{H} = (V, E)$ be a hb-graph with no repeated hb-edge. $V = \{v_i : i \in [\![n]\!]\}$.
  $\mathcal{H} = \bigoplus_{e \in E} \mathcal{H}_e$, $\mathcal{H}_e = (V, (e))$

Hb-graph polynomial

Iterative process on layers

- Each hb-graph can be summarized by a polynomial of degree $r_{\mathcal{H}}$:

$$P\left(\boldsymbol{z_0}\right) = \sum_{i=1}^{p} c_{e_i} P_{e_i}\left(\boldsymbol{z_0}\right)$$
$$= \sum_{i=1}^{p} c_{e_i} \frac{r_i!}{m_{ij_1}! \ldots m_{ij_{k_i}}!} q_{j_1^{m_{i\,j_1}},\ldots,j_{k_i}^{m_{i\,j_{k_i}}}} z_{j_1}^{m_{ij_1}} \ldots z_{j_{k_i}}^{m_{ij_{k_i}}}$$

- c_{e_i} is a technical coefficient => choosen to retrieve the m-degree of the vertices from the e-adjacency tensor.

m-uniform natural hb-graph

$\overline{k}$-adjacency hypermatrix

Let $\mathcal{H} = (V, E)$ be a r-m-uniform hb-graph. $V = \{v_i : i \in [\![n]\!]\}$.

- each hb-edge in a r-m-uniform hb-graph has same m-cardinality r
- **$\overline{k}$-adjacency hypermatrix of $\mathcal{H}$** is the hypermatrix $A_{\mathcal{H}} = (a_{i_1 \dots i_r})_{1 \leqslant i_1, \dots, i_r \leqslant n}$ defined by:

  $$A_{\mathcal{H}} = \sum_{i \in [\![p]\!]} Q_{e_i}$$

  where Q_{e_i} is the $\overline{k}$-adjacency hypermatrix of the elementary hb-graph associated to the hb-edge $e_i = \left\{ v_{j_1}^{m_{ij_1}}, \dots, v_{j_{k_i}}^{m_{ij_{k_i}}} \right\} \in E$.
- The only non-zero elements of Q_{e_i} are the elements of indices obtained by permutation of the multiset $\left\{ j_1^{m_{ij_1}}, \dots, j_{k_i}^{m_{ij_{k_i}}} \right\}$ and are all equals to $\dfrac{m_{ij_1}! \dots m_{ij_{k_i}}!}{(r-1)!}$.
- Remark: when the r-m-uniform hb-graph corresponds to a r-uniform hypergraph => retrieve the result of the degree-normalized tensor of Cooper and Dutle [2012].
- Claim: The m-degree of a vertex v_j in a r-m-uniform hb-graph $\mathcal{H}$ of $\overline{k}$-adjacency hypermatrix $\mathcal{A}_{\mathcal{H}}$ is:

  $$\deg_m(v_j) = \sum_{1 \leqslant j_2, \dots, j_r \leqslant n} a_{jj_2 \dots j_r}.$$

Elementary operations on hb-graphs I

Elementary operations

- Are needed for building the hb-graph m-uniformisation process
- **Canonical weighting operation:** $\phi_{\text{cw}} : \mathcal{H} = (V, E) \mapsto \mathcal{H}_1 = (V, E, w_1)$ where $\forall e \in E : w_1(e) = 1.$
- **c-dilatation operation:** $\phi_{\text{c-d}} : \mathcal{H}_1 = (V, E, w_1) \mapsto \mathcal{H}_c = (V, E, w_c)$ with $\forall e \in E : w_c(e) = c, c \in \mathbb{R}^{++}.$
- **y-complemented operation:** $\phi_{\text{y-c}} : \mathcal{H}_w = (V, E, w) \mapsto \tilde{\mathcal{H}}_{\tilde{w}} = \left(\tilde{V}, \tilde{E}, \tilde{w}\right)$, where $\tilde{\mathcal{H}}_{\tilde{w}}$ is the **y-complemented hbgraph** of $\mathcal{H}_w$ with:
  - $\tilde{V} = V \cup \{y\}$,
  - $\tilde{E} = (\xi(e))_{e \in E}$ where $\xi : E \to \mathcal{M}\left(\tilde{V}\right)$ is such that: $\forall e \in E$,

    $$\xi(e) = \left\{x^{m_{\xi(e)}(x)} : x \in \tilde{V}\right\} \in \mathcal{M}\left(\tilde{V}\right), \text{ with } m_{\xi(e)}(x) = \begin{cases} m_e(x) & \text{if } x \in e^\star \\ r_{\mathcal{H}} - \#_m e & \text{if } x = y \end{cases}$$
  - $\tilde{w}$ is such that $\forall e \in E$: $\tilde{w}(\xi(e)) = w(e)$.
- **Decomposition operation:** $\phi_{\text{d}} : \mathcal{H} \mapsto (\mathcal{H}_i)_{i \in I}$ such that: $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ where the $\mathcal{H}_i$ have all same universe and no pair of $\mathcal{H}_i$ have repeated hb-edge.

Elementary operations on hb-graphs II

Elementary operations

- **y^α-vertex-increasing operation:** $\phi_{y^\alpha\text{-v}} : \mathcal{H} = (V, E, w) \mapsto \mathcal{H}^+ = \left(V^+, E^+, w^+\right)$ where $\mathcal{H}^+$ is the
  - $V^+ = V \cup \{y\}$,
  - $E^+ = (\phi(e))_{e \in E}$ with the map $\phi : E \to \mathcal{M}\left(V^+\right)$ such that for all $e \in E$,
    $$\phi(e) = \left\{x^{m_{\phi(e)}(x)} : x \in V^+\right\} \in \mathcal{M}\left(V^+\right) \text{ with } m_{\phi(e)}(x) = \begin{cases} m_e(x) & \text{if } x \in e^\star \\ \alpha & \text{if } x = y \end{cases}$$
  - w^+ is such that $\forall e \in E$: $w^+(\phi(e)) = w(e)$.
- **Merging operation:** $\phi_{\text{m}} : (\mathcal{H}_a, \mathcal{H}_b) \mapsto \widehat{\mathcal{H}}$ where: $\widehat{\mathcal{H}_{\widehat{w}}} = \left(\widehat{V}, \widehat{E}, \widehat{w}\right)$ is the **merged hb-graph** of two weighted hb-graphs $\mathcal{H}_a = (V_a, E_a, w_a)$ and $\mathcal{H}_b = (V_b, E_b, w_b)$
  - $\widehat{V} = V_a \cup V_b$
  - $\widehat{E} = (\psi(e))_{e \in E_A + E_B}$ [6] where $\psi : E_A + E_B \to \mathcal{M}\left(\widehat{V}\right)$ such that for all $e \in E_A + E_B$, $\psi(e) = \left\{x^{m_{\psi(e)}(x)} : x \in \widehat{V}\right\} \in \mathcal{M}\left(\widehat{V}\right)$ with
    $$m_{\psi(e)}(x) = \begin{cases} m_e(x) & \text{if } x \in e^\star \\ 0 & \text{otherwise} \end{cases}$$
  - $\forall e \in E_a$, $\widehat{w}(e) = w_a(e)$ and $\forall e \in E_b$, $\widehat{w}(e) = w_b(e)$.

[6] $E_A + E_B$ is the family obtained with all elements of the family E_A and all elements of the family E_B

Elementary operations on hb-graphs III

e-adjacency and elementary operations

- Let $\mathcal{H} = (V, E)$ and $\mathcal{H}' = (V', E')$ be two hb-graphs. Let $\phi : \mathcal{H} \mapsto \mathcal{H}'$.
  - ϕ is said **preserving e-adjacency** if vertices of V' that are e-adjacent in $\mathcal{H}'$ are either e-adjacent vertices in $\mathcal{H}$ or the maximal subset of these vertices that are in V are e-adjacent in $\mathcal{H}$.
  - ϕ is said **preserving exactly e-adjacency** if vertices that are e-adjacent in $\mathcal{H}'$ are e-adjacent in $\mathcal{H}$ and reciprocally.
- **Claim 1:**
  - The composition of two operations which preserve (respectively exactly) e-adjacency preserves (respectively exactly) e-adjacency.
  - The composition of two operations where one preserves exactly e-adjacency and the other preverves e-adjacency preserves e-adjacency.
- **Claim 2:**
  - The canonical weighting operation, the c-dilatation operation, the merging operation and, the decomposition operation preserve exactly e-adjacency.
  - The y-complemented operation and the y^{α}-vertex-increasing operation preserve e-adjacency.

Processes involved for building the e-adjacency tensor

Processes used

- **Hb-graph m-uniformisation process** (Hm-UP): transform:
  - a hb-graph $\mathcal{H}$ of m-range $r_{\mathcal{H}}$
  - into a $r_{\mathcal{H}}$-m-uniform hb-graph written $\overline{\mathcal{H}}$
- **Polynomial homogenization process** (PHP): homogeneize the hb-graph polynomial

Choice of the technical coefficients

Principle

Choice of the c_{e_i} done such that the e-adjacency hypermatrix $A = (a_{i_1 \dots i_r})_{i_1,\dots,i_r \in [\![n]\!]}$ allows to retrieve:

- the m-degree of the vertices: $\sum\limits_{i_2,\dots,i_r \in [\![n]\!]} a_{i i_2 \dots i_r} = \deg_m(v_i)$.
- the number of hb-edges $|E|$.

Decomposition of a hb-graph

Principle of decomposition

- A hb-graph $\mathcal{H} = (V, E)$ decomposed in a family of r-m-uniform hb-graphs $(\mathcal{H}_r)_{r \in [\![r_\mathcal{H}]\!]}$.
- Quotienting the hb-edges:
  - $\mathcal{R}$ be the equivalency relation defined on E the family of hb-edges of $\mathcal{H}$: $e\mathcal{R}e' \Leftrightarrow \#_m e = \#_m e'$.
  - $E/\mathcal{R}$ is the set of classes of hb-edges of same m-cardinality. The elements of $E/\mathcal{R}$ are the sets: $E_r = \{e \in E : \#_m e = r\}$.
  - Considering $R = \{r : E_r \in E/\mathcal{R}\}$, it is set $E_r = \emptyset$ for all $r \in [\![r_\mathcal{H}]\!] \backslash R$.
  - $\mathcal{H}_r = (V, E_r)$ for all $r \in [\![r_\mathcal{H}]\!]$
  - =>

$$\mathcal{H} = \bigoplus_{r \in [\![r_\mathcal{H}]\!]} \mathcal{H}_r.$$

- Each $\mathcal{H}_r$ can be associated to a $\overline{k}$-adjacency tensor $\mathcal{A}_r$ viewed as a hypermatrix $A_{\mathcal{H}_r} = \left(a_{(r)i_1 \dots i_r}\right)$ of order r, hypercubic and, symmetric of dimension $|V| = n$.

Principle of conservation

Principle

- The number of hb-edges is kept constant in the decomposition of a hb-graph in layers.
- The processes involved in the uniformisation processes keep the number of hb-edges constant.
- $\sum_{i_1,\dots,i_{r_\mathcal{H}} \in [\![n_1]\!]} a_{i_1 \dots i_{r_\mathcal{H}}} = r_\mathcal{H} \left| E \right| .$
- $|E| = \sum_{r=1}^{n} |E_r| = \sum_{r=1}^{n} \frac{1}{r} \sum_{i_1,\dots,i_r \in [\![n]\!]} a_{(r) i_1 \dots i_r}$
- Hence, it follows: $\sum_{i_1,\dots,i_{r_\mathcal{H}} \in [\![n_1]\!]} a_{i_1 \dots i_{r_\mathcal{H}}} = \sum_{r=1}^{n} \frac{r_\mathcal{H}}{r} \sum_{i_1,\dots,i_r \in [\![n]\!]} a_{(r) i_1 \dots i_r} .$
- For all $r \in [\![r_\mathcal{H}]\!]$: $c_r = \frac{r_\mathcal{H}}{r}$. It is the technical coefficient for the corresponding layer of level r of the hb-graph $\mathcal{H}$.

Hm-UP I

Straightforward m-uniformisation

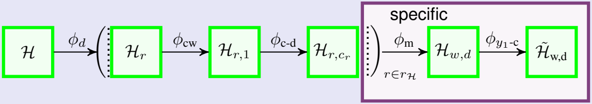


Figure 3: Operations on the original hb-graph to m-uniformize it in the straightforward approach. Parenthesis with vertical dots indicate parallel operations.

- The transformation $\phi_s : \mathcal{H} \mapsto \tilde{\mathcal{H}}_{w,d}$ preserves the e-adjacency.

Straightforward homogeneisation

- Transforming the hb-edge polynomial in a polynomial of degree $r_{\mathcal{H}}$ => details in article
- The hb-graph polynomial $P(z_0) = \sum_{i \in [\![p]\!]} c_i P_{e_i}(z_0)$ is transformed into a homogeneous polynomial:

$$R(\boldsymbol{z_1}) = \sum_{i \in [\![p]\!]} c_i R_{e_i}(\boldsymbol{z_1}) = \sum_{i \in [\![p]\!]} c_i z_{j_1}^{m_{i j_1}} \dots z_{j_{k_i}}^{m_{i j_{k_i}}} y_1^{m_{i\, n+1}}$$

representing the homogenized hb-graph $\overline{\mathcal{H}}$ with attached tensor $\mathcal{R} = \sum_{i=1}^{p} c_{e_i} \mathcal{R}_{e_i}$ where $c_{e_i} = \frac{r_{\mathcal{H}}}{\#_m e_i}$ and $m_{i\, n+1} = r_{\mathcal{H}} - \sharp_m e_i$.

PHP I (ii)

Straightforward e-adjacency hypermatrix

- The **straightforward e-adjacency hypermatrix of a hb-graph** $\mathcal{H} = (V, E)$ is the hypermatrix $\boldsymbol{A}_{\text{str},\mathcal{H}}$ defined by:

$$\boldsymbol{A}_{\text{str},\mathcal{H}} = \sum_{i \in [\![p]\!]} c_{e_i} \boldsymbol{R}_{\boldsymbol{e_i}}.$$

where for $e_i = \left\{ v_{j_1}^{m_{ij_1}}, \ldots, v_{j_{k_i}}^{m_{ij_{k_i}}} \right\} \in E$ the associated hypermatrix is:
$\boldsymbol{R}_{\boldsymbol{e_i}} = \left(r_{i_1 \ldots i_{r_\mathcal{H}}} \right)$, which only non-zero elements are:

$$r_{{j_1}^{m_{ij_1}} \ldots {j_{k_i}}^{m_{ij_{k_i}}} (n+1)^{m_{i\,n+1}}} = \frac{m_{ij_1}! \ldots m_{ij_{k_i}}! m_{i\,n+1}!}{r_\mathcal{H}!}$$

- with $m_{i\,n+1} = r_\mathcal{H} - \#_m e_i$ - and the ones with same value and obtained by permutation of the indices and where $c_{e_i} = \dfrac{r_\mathcal{H}}{\#_m e_i}$.

Hm-UP II

Silo m-uniformisation

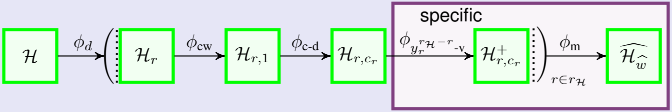


Figure 4: Operations on the original hb-graph to m-uniformize it in the silo approach. Parenthesis with vertical dots indicate parallel operations.

- **Claim:** The transformation $\phi_s : \mathcal{H} \mapsto \widehat{\mathcal{H}_{\widehat{w}}}$ preserves the e-adjacency.

Silo homogeneization

- $r_{\mathcal{H}} - 1$ additional variables respectively y_1 to $y_{r_{\mathcal{H}}-1}$.
- $P_{e_i}(\boldsymbol{z_0}) = z_{j_1}^{m_{ij_1}} \dots z_{j_{k_i}}^{m_{ij_{k_i}}}$ of $\deg P = \#_m e_i$.
- Add $y_{\#_m e_i}$ with multiplicity $m_{i\,\#_m e_i} = r_{\mathcal{H}} - \#_m e_i$ to have it of degree $r_{\mathcal{H}}$.
- $P_{e_i}(\boldsymbol{z_0})$ transformed in $R_{e_i}\left(\boldsymbol{z_{\#_m e_i}}\right) = P_{e_i}(\boldsymbol{z_0})\, y_{\#_m e_i}^{m_{i\,n+\#_m e_i}}$
- The only non-zero elements of $\mathcal{R}_{e_i}$ of rank $r_{\mathcal{H}}$ and dimension $n+1$ are:

$$r_{j_1^{m_{ij_1}} \dots j_{k_i}^{m_{ij_{k_i}}} (n+\#_m e_i)^{m_{i\,n+\#_m e_i}}} = \frac{m_{ij_1}! \dots m_{ij_{k_i}}! m_{i\,n+\#_m e_i}!}{r_{\mathcal{H}}!}$$

  and the one obtained by permutation of the indices of this first element.
- P is transformed into R the homogeneous polynomial attached to the homogeneised hb-graph $\overline{\mathcal{H}}$:

$$R\left(\boldsymbol{z_{r_{\mathcal{H}}-1}}\right) = \sum_{i \in [\![p]\!]} c_i R_{e_i}\left(\boldsymbol{z_{\#_m e_i}}\right) = \sum_{i \in [\![p]\!]} c_i z_{j_1}^{m_{ij_1}} \dots z_{j_{k_i}}^{m_{ij_{k_i}}} y_{\#_m e_i}^{m_{i\,n+\#_m e_i}}$$

- Attached tensor $\mathcal{R} = \sum_{i \in [\![p]\!]} c_{e_i} \mathcal{R}_{e_i}$ where: $c_{e_i} = \frac{r_{\mathcal{H}}}{\#_m e_i}$.

Silo e-adjacency hypermatrix

- The **silo e-adjacency hypermatrix of a hb-graph** $\mathcal{H} = (V, E)$ is the hypermatrix $\boldsymbol{A}_{\text{sil},\mathcal{H}} = \left(a_{i_1 \dots i_{r_\mathcal{H}}}\right)_{i_1,\dots,i_{r_\mathcal{H}} \in [\![n]\!]}$ defined by $\boldsymbol{A}_{\text{sil},\mathcal{H}} = \sum\limits_{i \in [\![p]\!]} c_{e_i} \boldsymbol{R}_{\boldsymbol{e_i}}$
  and where for $e_i = \left\{ v_{j_1}^{m_{ij_1}}, \dots, v_{j_{k_i}}^{m_{ij_{k_i}}} \right\} \in E$ the associated hypermatrix is:
  $\boldsymbol{R}_{\boldsymbol{e_i}} = \left(r_{i_1 \dots i_{r_\mathcal{H}}}\right)$, which only non-zero elements are:
  $$r_{{j_1}^{m_{ij_1}} \dots {j_{k_i}}^{m_{ij_{k_i}}} {(n+\#_m e_i)}^{m_{i\,n+\#_m e_i}}} = \frac{m_{ij_1}! \dots m_{ij_{k_i}}! m_{i\,n+\#_m e_i}!}{r_\mathcal{H}!}$$
  and all elements of $\boldsymbol{R}_{\boldsymbol{e_i}}$ obtained by permutation of the indices and with:
  $$m_{i\,n+\#_m e_i} = r_\mathcal{H} - \sum_{l \in [\![k_i]\!]} m_{i\,j_l},$$
  and where:
  $$c_{e_i} = \frac{r_\mathcal{H}}{\#_m e_i}.$$
- In this case, $\boldsymbol{A}_{\text{sil},\mathcal{H}} = \sum\limits_{r \in [\![r_\mathcal{H}]\!]} c_r \sum\limits_{e_i \in \{e : \#_m e = r\}} \boldsymbol{R}_{\boldsymbol{e_i}}$ where $c_r = \frac{r_\mathcal{H}}{r}$.

Hm-UP III

Layered m-uniformisation

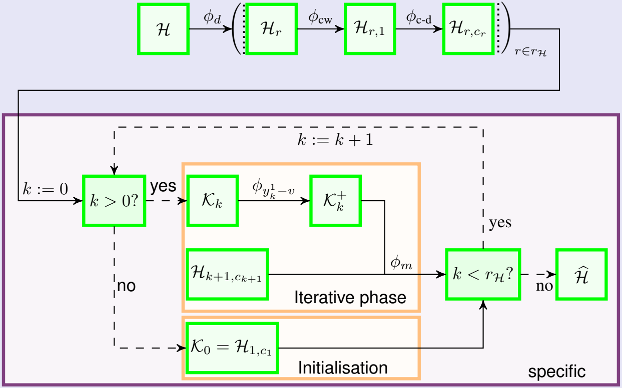


Figure 5: Operations on the original hb-graph to m-uniformize it in the layered approach. Parenthesis with vertical dots indicate parallel operations.

- **Claim:** The transformation $\phi_s : \mathcal{H} \mapsto \widehat{\mathcal{H}_{\widehat{w}}}$ preserves the e-adjacency.

PHP III (i)

Layered homogenization

- $r_{\mathcal{H}} - 1$ additional variables respectively y_1 to $y_{r_{\mathcal{H}}-1}$.
- $P_{e_i}(\boldsymbol{z_0}) = z_{j_1}^{m_{ij_1}} \dots z_{j_{k_i}}^{m_{ij_{k_i}}}$, $\deg P_{e_i}(z_0)$ is transformed into:
  $R_{(1)e_i}\left(\boldsymbol{z_{\#_m e_i}}\right) = P_{e_i}(\boldsymbol{z_0})\, y^1_{\#_m e_i}$ at the first iteration
  $R_{(k)e_i}\left(\boldsymbol{z_{\#_m e_i + k - 1}}\right) = P_{e_i}(\boldsymbol{z_0})\, y^1_{\#_m e_i} \dots y^1_{\#_m e_i + k - 1}$ at the k-th iteration
  $R_{(r_{\mathcal{H}} - \#_m e_i)e_i}\left(\boldsymbol{z_{r_{\mathcal{H}}-1}}\right) = P_{e_i}(\boldsymbol{z_0})\, y^1_{\#_m e_i} \dots y^1_{r_{\mathcal{H}}-1}$ at the last iteration
  Attached tensor $\mathcal{R}_{(r_{\mathcal{H}} - \#_m e_i)e_i}$ of rank $r_{\mathcal{H}}$ and dimension $n + r_{\mathcal{H}} - 1$.
- The only non-zero elements of $\mathcal{R}_{(r_{\mathcal{H}} - \#_m e_i)e_i}$ are:

$$r_{(r_{\mathcal{H}} - \#_m e_i)\, j_1^{m_{ij_1}} \dots j_{k_i}^{m_{ij_{k_i}}} [n + \#_m e_i]^1 \dots [n + r_{\mathcal{H}} - 1]^1} = \frac{m_{ij_1}! \dots m_{ij_{k_i}}!}{r_{\mathcal{H}}!}$$

  and the one obtained by permutation$R\left(\boldsymbol{z_{r_{\mathcal{H}}-1}}\right)$ is an homogeneous polynomial representing $\overline{\mathcal{H}}$
- $\mathcal{R} = \sum_{i \in [\![p]\!]} c_{e_i} \mathcal{R}_{(r_{\mathcal{H}} - \#_m e_i)e_i}$,where: $c_{e_i} = \dfrac{r_{\mathcal{H}}}{\#_m e_i}$.

PHP III (ii)

Layered homogenization

- The **layered e-adjacency tensor of a hb-graph** $\mathcal{H} = (V, E)$ is the tensor $\mathcal{A}_{\text{lay}}(\mathcal{H}) = \left(a_{i_1 \dots i_{r_\mathcal{H}}}\right)_{1 \leqslant i_1, \dots, i_{r_\mathcal{H}} \leqslant n}$ defined by:

$$\mathcal{A}_{\text{lay}}(\mathcal{H}) = \sum_{i \in \llbracket p \rrbracket} c_{e_i} \mathcal{R}_{(r_\mathcal{H} - \#_m e_i) e_i}$$

where for $e_i = \left\{ v_{j_1}^{m_{ij_1}}, \dots, v_{j_{k_i}}^{m_{ij_{k_i}}} \right\} \in E$ the associated tensor is:

$$\mathcal{R}_{(r_\mathcal{H} - \#_m e_i) e_i} = \left(r_{(r_\mathcal{H} - \#_m e_i) i_1 \dots i_{r_\mathcal{H}}} \right),$$

which only non-zero elements are:

$$r_{(r_\mathcal{H} - \#_m e_i) j_1^{m_{ij_1}} \dots j_{k_i}^{m_{ij_{k_i}}} [n + \#_m e_i]^1 \dots [n + r_\mathcal{H} - 1]^1} = \frac{m_{ij_1}! \dots m_{ij_{k_i}}!}{r_\mathcal{H}!}$$

and all elements of $\mathcal{R}_{e_i}$ with indices obtained by permutation and where: $c_{e_i} = \frac{r_\mathcal{H}}{\#_m e_i}$.

- $\mathcal{A}_{\text{lay}}(\mathcal{H})$ can also be written: $\mathcal{A}_{\text{lay}}(\mathcal{H}) = \sum_{r \in \llbracket r_\mathcal{H} \rrbracket} c_r \sum_{e_i \in \{e : \#_m e = r\}} \mathcal{R}_{e_i}$, where $c_r = \frac{r_\mathcal{H}}{r}$.

Spectral analysis of tensors

Eigenvalues

- $\lambda \in \mathbb{C}$ is an **eigenvalue** of $\mathcal{A}$ if it exists a nonzero vector $x \in \mathbb{C}^n$ such that:

$$\forall i \in [\![1, n]\!]\,,\ \left(\mathcal{A}x^{m-1}\right)_i = \lambda x_i^{m-1} \tag{1}$$

- x is called an **eigenvector** of $\mathcal{A}$ associated with the eigenvalue λ
- (x, λ) is called an **eigenpair** of $\mathcal{A}$.
- **spectrum of $\mathcal{A}$**: set of all eigenvalues of $\mathcal{A}$
- **spectra radius of $\mathcal{A}$**: $\rho(\mathcal{A})$: largest modulus of all eigenvalues
- **H-eigenvalue**:
  - eigenvalue λ of $\mathcal{A}$ with real eigenvector x associated to it.
  - x is called in this case an H-eigenvector.
- **Theorem:** Let $\mathcal{A} \in T_{m,n}$ be a nonnegative tensor. Then $\mathcal{A}$ has at least one H-eigenvalue and $\lambda_{H_{\max}}(\mathcal{A}) = \rho(\mathcal{A})$. Furthermore $\lambda_{H_{\max}}(\mathcal{A})$ has a non-negative H-eigenvector.

Spectral analysis of tensors

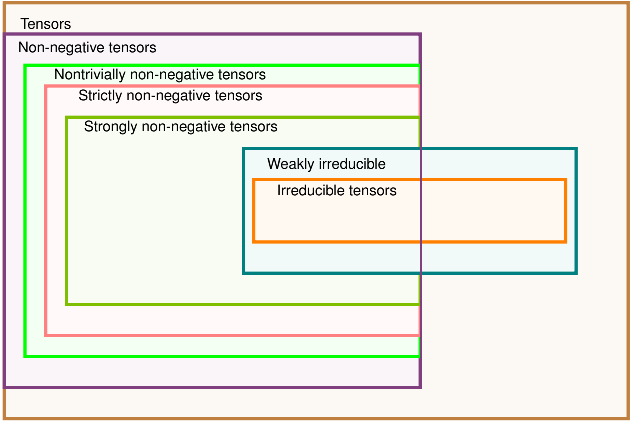


Figure 6: Different classes of tensors

Aim

Desirable that for constructed tensors:

- the spectral radius is positive
- there is a unique positive H-eigenvector (up to a multiplicative constant) associated to it

=> ensured for strongly non-negative tensor

Results on the constructed tensors

First spectral result

Claim: The e-adjacency tensor $\mathcal{A}_{\mathcal{H}} = \left(a_{i_1 \dots i_{k_{\max}}}\right)$ of a general hb-graph $\mathcal{H} = (V, E)$ has its eigenvalues λ such that:

$$|\lambda| \leqslant \max\left(\Delta, \Delta^{\star}\right) + r_{\mathcal{H}} \tag{2}$$

where $\Delta = \max\limits_{i \in \llbracket n \rrbracket} (d_i)$ and $\Delta^{\star} = \max\limits_{i \in \llbracket n_{\mathcal{A}} \rrbracket} (d_{n+i})$

Remark

- *In the straightforward approach:*
  $\Delta^{\star} = \deg_m(N_1) = \sum\limits_{j \in \llbracket r_{\mathcal{H}} - 1 \rrbracket} (r_{\mathcal{H}} - j) \left|\{e : \#_m e = j\}\right|$
- *In the silo approach:*
  $\Delta^{\star} = \max\limits_{j \in \llbracket r_{\mathcal{H}} - 1 \rrbracket} \left(\deg_m(N_j)\right) = \max\limits_{j \in \llbracket r_{\mathcal{H}} - 1 \rrbracket} \left((r_{\mathcal{H}} - j) \left|\{e : \#_m e = j\}\right|\right)$
- *In the layered approach:*
  $\Delta^{\star} = \max\limits_{j \in \llbracket r_{\mathcal{H}} - 1 \rrbracket} \left(\deg_m(N_j)\right) = \max\limits_{j \in \llbracket r_{\mathcal{H}} - 1 \rrbracket} \left(\left|\{e : \#_m e \leqslant j\}\right|\right) = \left|\{e : \#_m e \leqslant r_{\mathcal{H}} - 1\}\right|$
- *The values of Δ don't change whatever the approach taken is.*

Results on the constructed tensors

Classification of tensors

- <u>**Claim:**</u> Let $\mathcal{H} = (V, E)$ be a hb-graph which is not m-uniform and where $\bigcup_{e \in E} e^{\star} = V$.

  If $\mathcal{H}$ is connected then its straightforward e-adjacency tensor is symmetric nonnegative weakly irreducible.
- <u>Proof:</u> cf article
- To ensure weak irreducibility, the special vertex should be added to each hyperedge at least once.
- As a consequence, the spectral radius of this tensor is positive and associated to a unique positive Perron vector (up to a scaling factor)
- <u>**Claim:**</u> The three e-adjacency tensors built for hb-graphs are nontrivially nonnegative symmetric tensors when the hb-graph is connected and that the union of the support of hb-edges covers the vertex set.
- <u>Proof:</u> They have principal subtensors that are weakly irreducible, hence strictly non negative
- As a consequence the spectral radii of those tensors are positive.

Comparison I

Details

	$\mathcal{A}_{\text{str}}(\mathcal{H})$	$\mathcal{A}_{\text{sil}}(\mathcal{H})$	$\mathcal{A}_{\text{lay}}(\mathcal{H})$
Order	$r_{\mathcal{H}}$	$r_{\mathcal{H}}$	$r_{\mathcal{H}}$
Dimension	$n+1$	$n+r_{\mathcal{H}}-1$	$n+r_{\mathcal{H}}-1$
Total number of elements	$(n+1)^{r_{\mathcal{H}}}$	$(n+r_{\mathcal{H}}-1)^{r_{\mathcal{H}}}$	$(n+r_{\mathcal{H}}-1)^{r_{\mathcal{H}}}$
Total number of elements potentially used by the way the tensor is build	$(n+1)^{r_{\mathcal{H}}}$	$(n+r_{\mathcal{H}}-1)^{r_{\mathcal{H}}}$	$(n+r_{\mathcal{H}}-1)^{r_{\mathcal{H}}}$
Number of repeated elements per hb-edge $e_j = \left\{ v_{i_1}^{m_{ji_1}}, \dots, v_{i_j}^{m_{ji_j}} \right\}$	$\frac{r_{\mathcal{H}}!}{m_{ji_1}! \dots m_{ji_j}! n_j!}$ with $n_j = r_{\mathcal{H}} - \#_m e_j$	$\frac{r_{\mathcal{H}}!}{m_{ji_1}! \dots m_{ji_j}! n_{jk}!}$ with $n_{jk} = r_{\mathcal{H}} - \#_m e_j$	$\frac{r_{\mathcal{H}}!}{m_{ji_1}! \dots m_{ji_j}!}$
Number of elements to be filled per hyperedge of size s before permutation	Constant 1	Constant 1	Constant 1
Number of elements to be described to derived the tensor by permutation of indices	$\lvert E \rvert$	$\lvert E \rvert$	$\lvert E \rvert$

Comparison II

Details

	$\mathcal{A}_{\text{str}}(\mathcal{H})$	$\mathcal{A}_{\text{sil}}(\mathcal{H})$	$\mathcal{A}_{\text{lay}}(\mathcal{H})$
Value of elements of a hyperedge	Dependent of hb-edge composition $\frac{m_{ji_1}!\ldots m_{ji_j}!n_j!}{(r_{\mathcal{H}}-1)!}$	Dependent of hb-edge composition $\frac{m_{ji_1}!\ldots m_{ji_j}!n_{jk}!}{(r_{\mathcal{H}}-1)!}$	Dependent of hb-edge composition $\frac{m_{ji_1}!\ldots m_{ji_j}!}{(r_{\mathcal{H}}-1)!}$
Symmetric	Yes	Yes	Yes
Reconstructivity	Straightforward: delete special vertices	Straightforward: delete special vertices	Straightforward: delete special vertices
Nodes degree	Yes, but not straightforward	Yes	Yes
Spectral analysis	Special vertex increases the amplitude of the bounds	Special vertices increase the amplitude of the bounds	Special vertices increase the amplitude of the bounds
Interpretability of the tensor in term of hb-graph	Yes	Yes	Yes

e-adjacency tensor of a hypergraph

Definition

The e-adjacency tensor of a hypergraph $\mathcal{H} = (V, E)$ having maximal cardinality of its hyperedges $k_{\max}$ is the tensor $\mathcal{A}(\mathcal{H}) = \left(a_{i_1 \dots i_{r_\mathcal{H}}}\right)_{1 \leqslant i_1, \dots, i_{r_\mathcal{H}} \leqslant n}$ defined by:

$$\mathcal{A}(\mathcal{H}) = \sum_{i \in [\![p]\!]} c_{e_i} \mathcal{R}_{e_i}$$

and where for $e_i = \left\{v_{j_1}, \dots, v_{j_{k_i}}\right\} \in E$ the associated tensor is: $\mathcal{R}_{e_i} = \left(r_{i_1 \dots i_{r_\mathcal{H}}}\right)$, which only non-zero elements are:

$$r_{j_1 \dots j_{k_i} (n+k_i)^{k_{\max} - k_i}} = \frac{(k_{\max} - k_i)!}{k_{\max}!}$$

and all elements of $\mathcal{R}_{e_i}$ obtained by permuting

$$j_1 \dots j_{k_i} (n + k_i)^{k_{\max} - k_i} ,$$

and where:

$$c_{e_i} = \frac{k_{\max}}{k_i}.$$

Hypergraph e-adjacency tensor

Comparison with existing tensor

	$\mathcal{B}_{\mathcal{H}}$	$\mathcal{S}_{\mathcal{H}}$	$\mathcal{A}(\mathcal{H})$
Order	$k_{\max}$	$k_{\max}$	$k_{\max}$
Dimension	n	n	$n + k_{\max} - 1$
Total number of elements	$n^{k_{\max}}$	$n^{k_{\max}}$	$(n + k_{\max} - 1)^{k_{\max}}$
Total number of elements potentially used by the way the tensor is build	$n^{k_{\max}}$	$n^{k_{\max}}$	$(n + k_{\max} - 1)^{k_{\max}}$
Number of non-nul elements for a given hypergraph	$\sum\limits_{s=1}^{k_{\max}} \alpha_s \|E_s\|$ with $\alpha_s = p_s(k_{\max}) \frac{k_{\max}!}{k_1!\ldots k_s!}$	$\sum\limits_{s=1}^{k_{\max}} s! \|E_s\|$	$\sum\limits_{s=1}^{k_{\max}} \alpha_s \|E_s\|$ with $\alpha_s = \frac{k_{\max}!}{k_1!\ldots k_s! n_s!}$ with $n_s = k_{\max} - s$
Number of repeated elements per hyperedge of size s	$\frac{k_{\max}!}{k_1!\ldots k_s!}$	$s!$	$\frac{k_{\max}!}{k_1!\ldots k_s! n_s!}$ with $n_s = k_{\max} - s$
Number of elements to be filled per hyperedge of size s before permutation	Varying $p_s(k_{\max})$	Varying s if prefix is considered as nonpermuting part	Constant 1
Number of elements to be described to derived the tensor by permutation of indices	$\sum\limits_{s=1}^{k_{\max}} p_s(k_{\max}) \|E_s\|$	$\sum\limits_{s=1}^{k_{\max}} s \|E_s\|$	$\|E\|$

Hypergraph e-adjacency tensor

Comparison with existing tensor

	$\mathcal{B}_{\mathcal{H}}$	$\mathcal{S}_{\mathcal{H}}$	$\mathcal{A}(\mathcal{H})$
Value of elements of a hyperedge	Dependent of hyperedge composition $\frac{s}{\alpha_s}$	Dependent of hyperedge composition $\frac{1}{(s-1)!}$	Dependent of hyperedge size $\frac{(k_{\max}-s)!}{s\,(k_{\max}-1)!}$
Reconstructivity	Need computation of duplicated vertices	Need computation of duplicated vertices	Straightforward: delete special vertices
Nodes degree	Yes	Yes	Yes
Spectral analysis	Yes	Yes	Special vertices increase the amplitude of the bounds
Classification	Nonnegative, symmetric, weakly irreducible if hypergraph connected and $\bigcup\limits_{e\in E} e = V$	Nonnegative, weakly irreducible if hypergraph connected and $\bigcup\limits_{e\in E} e = V$	Nonnegative, symmetric, nontrivially nonnegative
Interpretability of the tensor in term of hypergraph / hb-graph	No / No	No / No	No / Yes

Several questions remain

- What is the impact of the proposed tensors on spectral hypergraph theory?
- What is hb-graph spectral theory?
- Can we use hb-graph e-adjacency tensor for diffusion?
- What is the gain of moving from the incident matrix to the e-adjacency hypermatrix?

Thank you for your attention !

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Multisets IV

Given: a natural multiset $A_m = (A, m)$ of universe $A = \{\alpha_i : i \in [\![n]\!]\}$ and multiplicity function m.

An other definition

- Definition in Syropoulos [2000]: $< A_0, \rho >$
  - A_0 is the set of all instances (including copies) of A_m
  - Equivalency relation ρ where:

$$\forall x \in A_0, \forall x' \in A_0 : \; x\rho x' \Leftrightarrow \exists! c \in A : x\rho c \wedge x'\rho c.$$

- Also A_0/ρ is isomorphic to A
- $\forall \overline{x} \in A_0/\rho, \exists! c \in A : |\{x : x \in \overline{x}\}| = m(c) \wedge \forall x \in \overline{x} : x\rho c.$
- A_0 is called a **copy-set** of the multiset A_m.

Remark

- *A copy-set for a given multiset is not unique.*
- *Sets of equivalency classes of two couples $< A_0, \rho >$ and $< A'_0, \rho' >$ of a given multiset are isomorphic.*

Numbered-copy-hypergraph

Numbering copies

- In natural hb-graphs:
  - vertices in a hb-edge with multiplicities greater than 1 = copies of the original vertex
  - idea: numbering the copies
- **Numbered copy-set** of a natural multiset $A_m = \left\{ x_i^{m_i} : i \in [\![n]\!] \right\}$ is the copy-set $\breve{A_m} = \left\{ [x_{i\,j}]_{m_i} : i \in [\![n]\!] \right\}$ where:
  - $[x_{i\,j}]_{m_i} = \left\{ x_{i\,1}, ..., x_{i\,m_i} \right\}$ copies of x_i
  - j = copy number of the element x_i.
- **Maximum multiplicity function of a hb-graph**: $\forall v \in V : m(v) = \max\limits_{e \in E} m_e(v)$.
- **Numbered-copy-hypergraph** of $\mathcal{H}$: $\mathcal{H}_0 = \left(\breve{V}, E_0 \right)$ where $E_0 = \{ e_{k\,0} : \; k \in [\![p]\!] \}$:
  - Vertex set: Numbered-copy-set of the multiset $\left\{ v_i^{m(v_i)} : i \in [\![n]\!] \right\}$:

    $\breve{V} = \left\{ [v_{i\,j}]_{m(v_i)} : i \in [\![n]\!] \right\}$.
  - Each hb-edge $e_k = \left\{ v_{i_j}^{m_{k\,i_j}} : j \in [\![k]\!] \wedge i_j \in [\![n]\!] \right\}$ is associated to a copy-set / equivalency relation $< e_{k\,0}, \rho_k >$ which elements are in $\breve{V}$ with copy number as small as possible for each vertex in e_k.
- Claim: A numbered-copy-hypergraph is unique for a given hb-graph.